Math 3 - Discrete math (V3020) **Exercises, week 2 (due Sep 16th 2025)**

1. Show that every positive integer n has a unique expression of the form $n = 2^r m$, where $r \geqslant 0$ and m is a positive odd integer.

2. Show that if $x, y \in \mathbb{Z}$ are odd, then $x^2 + y^2$ cannot be a perfect square. (Hint: One of the exercises from week 1 can be a helpful starting point.)

3. Show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

for any integer $n \geqslant 0$.

4. Let $f(x)$ and $g(x)$ be functions that are n times differentiable. Show that the nth differential of the product $f(x)g(x)$ has the form

$$\frac{d^n}{dx^n}(f(x)g(x)) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

Here we have used the notation for the kth derivative $\frac{d^k}{dx^k} f(x) = f^{(k)}(x)$. (Hint: This is the generalization of the familiar product rule $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$. Use induction. Some of the intermediate steps in the final induction step can be tricky, but here Pascal's triangle could be useful to see some connections between the terms.)