Math 3 (V3020) — Exercises, week 2

1. Claim: Every positive integer n has a unique expression of the form $n = 2^r m$, $r \geq 0$, m a positive odd integer.

Proof: Write $n = \prod_p p^{\alpha(p)}$, where p go over the prime numbers, and $\alpha(p) = 0$ for large enough primes.

Now take the first prime, 2, out of the product:
$n = 2^{\alpha(2)} \prod_{p>2} p^{\alpha(p)}$
and notice that all primes $p > 2$ are odd.

We notice that the product of any two odd integers
$(2k+1)(2m+1) = 4km + 2k + 2m + 1 = 2(km + k + m) + 1$
is odd, so $\prod_{p>2} p^{\alpha(p)}$ has to be odd.

The uniqueness of $n = 2^{\alpha(2)} \prod_{p>0} p^{\alpha(p)} =: 2^r m$, where

m is odd, follows from the uniqueness of the prime decomposition (the fundamental theorem of arithmetic).

(2.) Claim: If $x, y \in \mathbb{Z}$ are odd, then x^2+y^2 cannot be a perfect square.

Proof:
$x, y \in \mathbb{Z}$ are odd, so denote $x = 2m+1$ and $y = 2n+1$, $m, n \in \mathbb{Z}$.

Now

$$x^2+y^2 = (2m+1)^2+(2n+1)^2 = 4(m^2+n^2)+4(m+n)+2$$
$$= 2[2(m^2+n^2+m+n)+1]$$

which is an even number. However $2(m^2+n^2+m+n)+1$ is odd, and we used this fact to show $4 = 2^2 \nmid x^2+y$ in the last week's exercises.

Let us construct a proof by contradiction: If we assumed x^2+y^2 was a perfect square, then

$$x^2+y^2 = k^2 = \prod_p p^{2\alpha(p)}$$

for some $k = \prod_p p^{\alpha(p)} \in \mathbb{Z}$.

Because x^2+y^2 is even, we would have to have 2 as its prime factor:

$$x^2+y^2 = 2^{2\alpha(2)} \prod_{p>2} p^{2\alpha(p)},$$

where $\alpha(2) \geq 1$, meaning $4 \mid x^2+y$.

However, we know that $4 \nmid x^2+y^2$ which is a contradiction.

$\square$

(3.) Claim : $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. (Hint: use the binomial theorem.)

Proof:

From the form of the sum it is implicit that $0 \leq k \leq n$. Therefore $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The binomial theorem states for any $n \geq 1$, $x, y \in \mathbb{R}$
$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

The case $n = 0$ can be checked separately
$\sum_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = \frac{0!}{0!(0-0)!} = 1 = 2^0$ OK!

Now choose $x = y = 1$ for the $n \geq 1$ case and use the binomial theorem
$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1+1)^n = 2^n$.

$\square$

4. Claim: Let $f(x), g(x)$ be n times differentiable functions. Then the nth differential of $f(x)g(x)$ has the form

$$\frac{d^n}{dx^n}\left(f(x)g(x)\right) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x)$$

where we use the notation $\frac{d^k}{dx^k} f(x) = f^{(k)}(x)$.

Proof:

Let us prove the claim by induction.

Check the case $n=1$:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$
$$= \sum_{k=0}^{1} \binom{1}{k} f^{(k)}(x)\, g^{(1-k)}(x) \qquad \text{OK.}$$

Induction assumption: The claim is true for $n=m$, i.e. $\frac{d^m}{dx^m}\left(f(x)g(x)\right) = \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(x)\, g^{(m-k)}(x)$

Induction step:

Check the case $n = m+1$. Denote $f(x) = f$.

$$\begin{aligned}
\frac{d^{m+1}}{dx^{m+1}}(fg) &= \frac{d}{dx}\left[\frac{d^m}{dx^m}(fg)\right] \\
&= \frac{d}{dx}\left[\sum_{k=0}^{m}\binom{m}{k} f^{(k)} g^{(m-k)}\right] \\
&= \sum_{k=0}^{m}\binom{m}{k} f^{(k)} g^{(m+1-k)} + \sum_{k=0}^{m}\binom{m}{k} f^{(k+1)} g^{(m-k)}
\end{aligned}$$

The summation index is just a dummy index

so we can rename $k+1 = l$ in

$$\begin{aligned}
\sum_{k=0}^{m}\binom{m}{k} f^{(k+1)} g^{(m-k)} &= \sum_{l=1}^{m+1}\binom{m}{l-1} f^{(l)} g^{(m+1-l)} \\
&= \sum_{k=1}^{m+1}\binom{m}{k-1} f^{(k)} g^{(m+1-k)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{d^{m+1}}{dx^{m+1}}(fg) &= \binom{m}{0} f^{(0)} g^{(m+1)} + \sum_{k=1}^{m}\binom{m}{k} f^{(k)} g^{(m+1-k)} \\
&\quad + \sum_{k=1}^{m}\binom{m}{k-1} f^{(k)} g^{(m+1-k)} + \binom{m}{m} f^{(m+1)} g^{(0)}
\end{aligned}$$

Note that $\binom{m}{0} = \binom{m+1}{0}$ and $\binom{m}{m} = \binom{m+1}{m+1}$

$$\begin{aligned}
\Rightarrow \frac{d^{m+1}}{dx^{m+1}}(fg) &= \binom{m+1}{0} f^{(0)} g^{(m+1)} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(k)} g^{(m+1-k)} \\
&\quad + \binom{m+1}{m+1} f^{(m+1)} g^{(0)}
\end{aligned}$$

PASCAL'S TRIANGLE IDENTITY

$$\begin{aligned}
&= \binom{m+1}{0} f^{(0)} g^{(m+1)} + \sum_{k=1}^{m}\binom{m+1}{k} f^{(k)} g^{(m+1-k)} + \binom{m+1}{m+1} f^{(m+1)} g^{(0)} \\
&= \sum_{k=0}^{m+1}\binom{m+1}{k} f^{(k)} g^{(m+1-k)}
\end{aligned}$$

$\square$