Math 3 (V3020) Exercises, week 5

1. Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k and e an edge of G.

How many vertices and edges are in

a) $G-e$?
We only remove one edge, e, from G, so there are n vertices and $m-1$ edges in $G-e$.

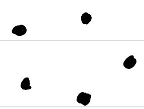
b) $G-v$?
We remove the vertex v and the k incident edges from G, so there are $n-1$ vertices and $m-k$ edges in $G-v$.

c) $G\setminus e$?
Edge e is contracted (removed and end vertices identified as one) $\Rightarrow$ $n-1$ vertices and $m-1$ edges remain in $G\setminus e$.

2. Draw the following graphs

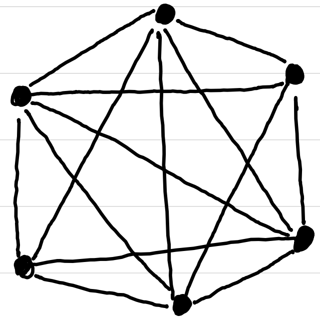
a) the null graph N_5

The edge set of a null graph is empty, so

 is a null graph with 5 vertices, N_5.

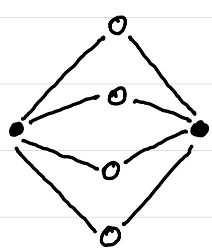
b) the complete graph K_6

This is a simple graph in which each pair of the distinct (6) vertices are adjacent.



c) The complete bipartite graph $K_{2,4}$

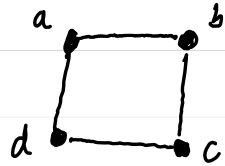
A complete bipartite graph is a bipartite graph (can be split to disjoint sets A and B so that each edge joins a vertex from A to a vertex in B) where there is only one vertex joining each vertex of A to each vertex of B.



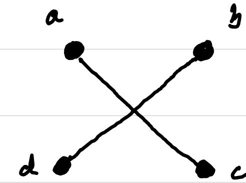
d) the complement of the cycle graph C_4

A complement of a simple graph G with vertex set $V(G)$, is the graph $\bar{G}$ with vertex set $V(G)$ in which two vertices are adjacent if and only if the are not adjacent in G.

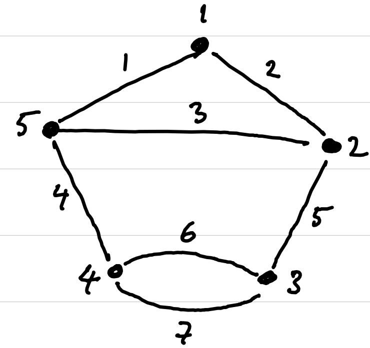
C_4 :



$\bar{C}_4$:



3. Write the adjacency and incidence matrix of



The ij-th entry of the adjacency matrix is the number of edges joining vertices i and j.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The ij-th entry of the incidence matrix is 1 if vertex i is incident to edge j, and 0 otherwise

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

4. Are the following statements true or false? Why?

a) Any two isomorphic graphs have the same degree sequence.

The claim is <u>true</u>. Proof:

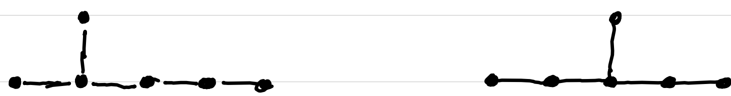
Let G_1 and G_2 be isomorphic graphs. This means that there is a one-to-one correspondence between the vertices of G_1 and G_2 s.t. the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2.

In other words, for every vertex v_i of G_1 with, there is a corresponding vertex u_j of G_2 with an equal number of adjacent vertices, i.e. $\deg(v_i) = \deg(u_j)$.

Therefore the degree sequences, that are just the degrees of every vertex listed in ascending order, must match.

b) Any two graphs with the same degree sequence are isomorphic.

The claim is <u>false</u> due to the following counter example:



These graphs have the degree sequence $(1, 1, 1, 2, 2, 3)$, but they are not isomorphic.

5. A simple graph that is isomorphic to its complement is called self-complementary.

Claim: If G is self-complementary, then G has $4k$ or $4k+1$ vertices, $k \in \mathbb{N}$.

Proof: Let G be a self-complementary graph.
(Two graphs G_1 and G_2 are isomorphic if there is a one-to-one correspondence between vertices of G_1 and G_2 s.t. the number of edges joining any vertices in G_1 equals the number of them joining the corresponding vertices in G_2.)

Notice that a complement of a graph G with vertex set $V(G)$, (n total vertices), contains the edges that are in the complete graph with vertices $V(G)$, but not in the edge set of G, $E(G)$.

In other words, we obtain $\bar{G} = K_n - E(G)$, where K_n is the complete graph with n vertices, given that G has n vertices.

Since G is isomorphic to $\bar{G}$, they have the same degree sequences. Therefore, there is an equal amount of vertices in G and $\bar{G}$, and an equal amount of edges in G and $\bar{G}$.

Let G have n vertices.
Since a complete graph K_n has $n(n-1)/2$ edges, and a self-complementary G and its complement must have the same amount of edges, this number has to be half of that,
$[n(n-1)/2]/2 = n(n-1)/4$.

This number naturally has to be an integer, so the number of vertices, n, has to be such that the following is true.

$\rightarrow 4 \mid n(n-1) \Longleftrightarrow n(n-1) = 4m$

Since n and $n-1$ cannot both be divisible by 2, one of them has to be divisible by 4
$\Rightarrow n = 4k$ or $n-1 = 4k$, i.e $n = 4k+1$ for $k \in \mathbb{N}$.