Math 3 (V3020) Exercises, week 6

1. Prove the "$\Leftarrow$" direction of Theorem 3.2.3, i.e if every cycle of G has an even length, then G is a bipartite graph.

Proof: Let $v_0 \to v_1 \to \dots \to v_m \to v_0$ be a cycle in G, so it has an even length. Therefore, we can color the vertices so that v_0 is white, v_1 black, v_2 is white, ... v_m is black, i.e. with alternating colors. If this is the only cycle of G, G is bipartite.

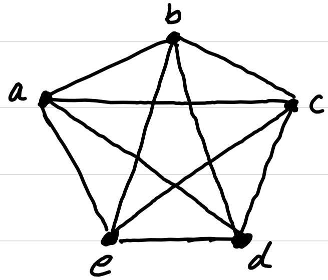
Assuming there are more than one cycles that involve at least one of the vertices of the above cycle, say v_0. Let one of these cycles be $v_0 \to u_1 \to \dots \to u_n \to v_0$. Due to being of even length, also the vertices of this cycle can be colored with alternating colors, v_0 still being white.

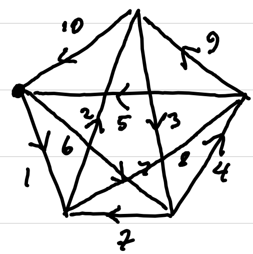
Now v_0, a white vertex, is still adjacent only to black vertices. The same process can be repeated for every other vertex that is also part of multiple cycles.

Therefore G is bipartite.

2. Are the following graphs Eulerian or semi-Eulerian?

a) Complete graph K_5?





Let us take the following walk:

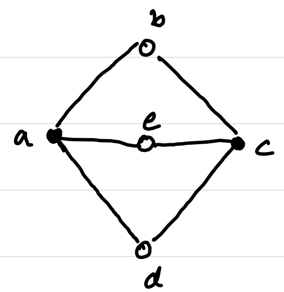
$a \to e \to b \to d \to c \to a$
$\to d \to e \to c \to b \to a$

Here every edge is distinct, so the walk is a trail, and it start and ends with the same vertex, so it is closed.

Since K_5 is connected and the closed trail that we found contains every edge in K_5, K_5 is Eulerian.

We could also use Theorem 3.2.9 and notice that every vertex of K_5 has even degree, 4, giving that K_5 is Eulerian.

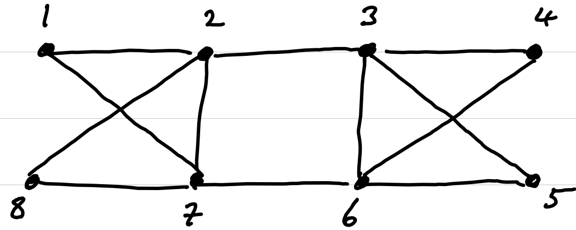
b) The complete bipartite graph $K_{2,3}$?



Immediately Theorem 3.2.9. gives us that $K_{2,3}$ is not Eulerian because $\deg(a) = \deg(c) = 3$ are odd.

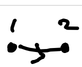
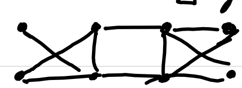
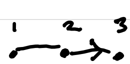
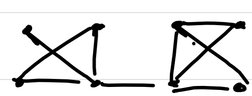
However, the trail $a \to b \to c \to e \to a \to d \to c$ contains every edge of $K_{2,3}$ (but is not closed), making $K_{2,3}$ semi-Eulerian.

3. Use Fleury's algorithm to produce an Eulerian trail for the graph below



The degree of every vertex is even, so the graph is Eulerian.

We traverse edges, starting from vertex 1 and erase them and isolated vertices as we go. We use bridges when there is no alternative

Edge	Trail so far	Remaining graph
12		
23		
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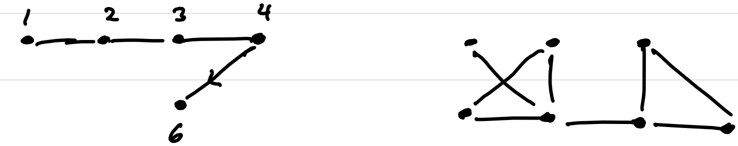

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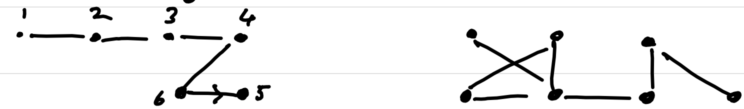

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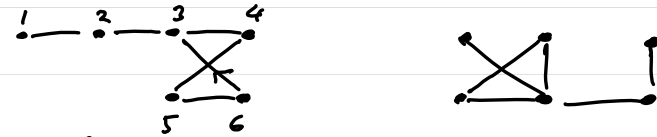

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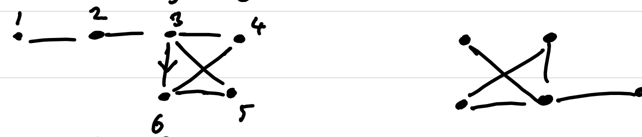

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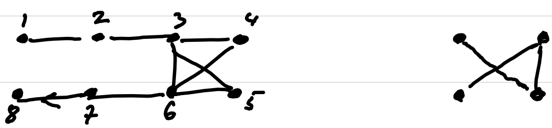

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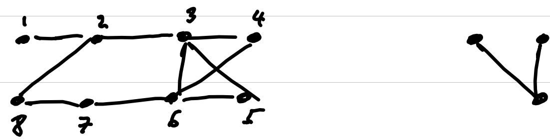

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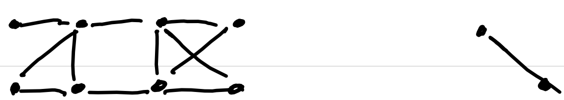

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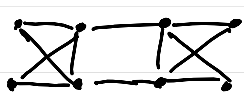
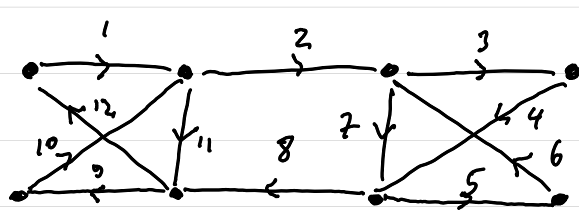
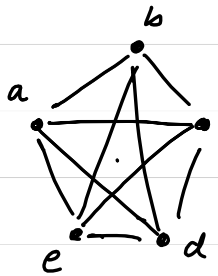




4. Are the following graphs Hamiltonian or semi-Hamiltonian?

a) The complete graph K_5?

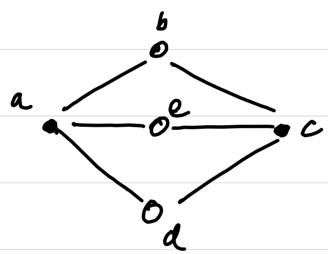


The closed trail

$a \to b \to c \to d \to e \to a$

passes through every vertex of K_5 once. Therefore K_5 is Hamiltonian.

b) The complete bipartite graph $K_{2,3}$?



Clearly $K_{2,3}$ is not Hamiltonian because every closed trail going through every vertex would have to pass through a or c more than once.

However $e \to a \to b \to c \to d$ passes once through every vertex but is not closed so $K_{2,3}$ is semi-Hamiltonian.