Math (V3020) Exercise, week 7

(1.) Prove that every tree is a bipartite graph.

Let T be a tree with n vertices. Theorem 3.3.2 tells us that this is equivalent with the following two statements. T is connected and each edge is a bridge; Any two vertices of T are connected by exactly one path.

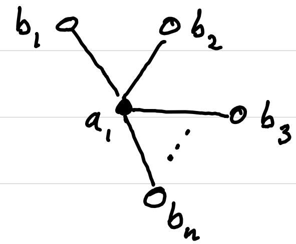
We can now choose any vertex of T, call it v_0. Let v_0 be part of the set A. Now every adjacent vertex is connected to v_0 exactly by one path (the single bridge), so we can choose these vertices to be in set B. We carry on from these vertices similarly. Every adjacent vertex is part of the set A in turn, because the vertices of this initial set B are connected through a path via v_0.

Since T is connected, we are able to carry on this alternating process and cover all vertices of T. The result is that vertices in set A are only adjacent to vertices in set B, and vice versa, making T bipartite.

2. Which trees are complete bipartite graphs?

A complete bipartite graph is a bipartite graph where each vertex of set A is joined to each vertex in B by just one edge.

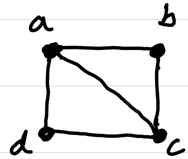
Let us construct a complete bipartite tree. An initial vertex a_1, in set A, is adjacent to n vertices b_i in set B



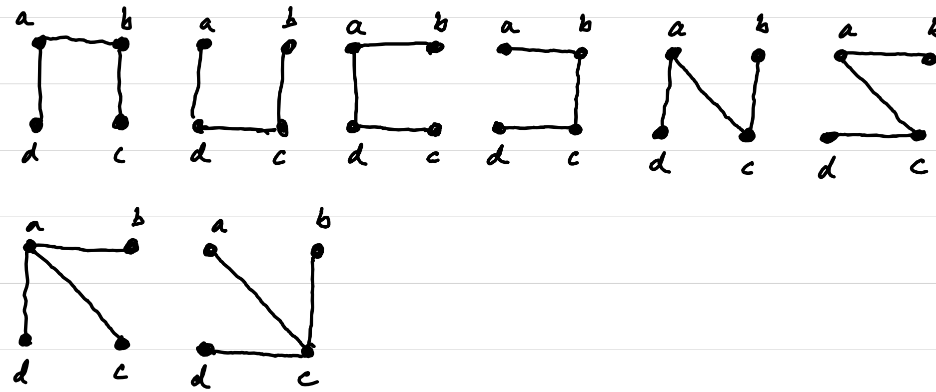
The addition of any vertex in set A and keeping the requirement of a complete bipartite graph (this new vertex a_2 is adjacent to each b_i) would form a cycle, thus making the graph no longer a tree.

Only trees $K_{1,n}$ are bipartite graphs.

3. Draw every spanning tree of the following graph



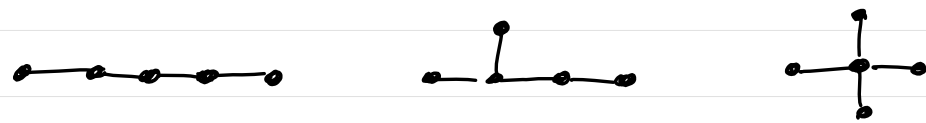
In other words, we want to construct every tree that contains all vertices of this graph.



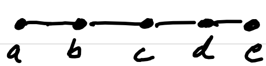
(There was some confusion when we looked at the solution during the lecture. There we drew couple of extra spanning trees that were wrong by accident. They would have been right if there was an edge bd. The ones shown above are the only correct ones for the graph in the actual exercise.)

4. Verify directly that there are exactly 125 labeled trees of 5 vertices.

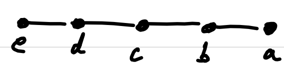
There a three unlabeled trees of 5 vertices:



The first one can be labeled $\frac{5!}{2} = 60$ ways. The division by 2 comes from the fact that rotating the graph doesn't change the labeling. For example,

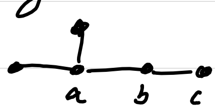


and



are the same.

The second graph can also be labeled in 60 ways by choosing $5 \cdot 4 \cdot 3 = 60$ ways of labeling a, b, c in



There are only 5 ways of labeling the third graph, i.e. the 5 ways of choosing the "middle" vertex.

So, in total, there are $60 + 60 + 5 = 125$ labelings.